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Convergence Estimates for Crude Approximations of a Pareto Set

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Abstract—The crude global search that is used in parameter space investigation provides approximations to the set of efficient points. Estimates of approximation errors are established and numerical examples confirm these estimates. The investigated examples can be used as tests for various numerical methods.

Keywords—Multiple criteria decison making, Pareto-optimality, Efficient points, Nondominated points, Optimization, Quasi-random search.

1. INTRODUCTION

Consider a nonlinear problem with multiple objectives of the form

$$f_1(x) \to \min, \dots, f_k(x) \to \min; \qquad x \in I^n, \tag{1}$$

where $x = (x_1, \dots, x_n)$ is a point of the n-dimensional unit cube I^n, so that $0 \le x_j \le 1$, $1 \le j \le n$.

The parameter space investigation (PSI) method was developed [1,2] as an interactive method for constructing a set of admissible solutions (that is, solutions with acceptable objective function values for all objectives simultaneously). However, PSI can also be used for obtaining approximations to the set E of efficient (or nondominated or Pareto-optimal) points. Its algorithm is easy to handle and very reliable though the convergence is, in general, slow.

In [3], the bi-criterial case ($k = 2$) was discussed. Here, the general multicriterial case is considered and approximation errors are estimated. If N trial points are used in PSI, the general convergence rate is $N^{-1/n}$, but it may be better in particular cases.

2. COMPUTATION ALGORITHM

Trial points $x^{(1)}, x^{(2)}, \dots$ are selected that fill uniformly the cube. At each of these points, say $x^{(i)}$, the criterion vector $(f_1(x^{(i)}), \dots, f_k(x^{(i)}))$ is computed. If this vector is dominated by any currently retained criterion vector, the point $x^{(i)}$ is discarded. If not, the point $x^{(i)}$ is retained while all currently retained points with criteria vectors dominated by the new one must be discarded.

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Thus, for $i = N$, we obtain $N_0 \leq N$ trial points that are called *approximately efficient*. The finite set of these N_0 points is denoted by E_N and will be regarded as an approximation to E.

In PSI, the decision variables $x_1, \ldots, x_n$ are often called parameters. Hence, the name of the method.

3. OBJECTIVE FUNCTIONS SATISFYING A GENERAL LIPSCHITZ CONDITION

Assume that all the functions $f_p(x)$, $1 \leq p \leq k$, satisfy a common general Lipschitz condition: for arbitrary points x and x' in I^n

$$|f_p(x) - f_p(x')| \leq \rho(x, x'), \tag{2}$$

where

$$\rho(x, x') = \sum_{j=1}^{n} L_j |x_j - x'_j|$$

and all $L_j \geq 0$. The word "general" is used to stress that the Lipschitz constants L_j may be different, some L_j may even be zero.

In [4], the ρ-dispersion d_ρ of the points $x^{(1)}, \ldots, x^{(N)}$ was introduced,

$$d_\rho = \sup_{x \in I^n} \min_{1 \leq i \leq N} \rho\left(x, x^{(i)}\right). \tag{3}$$

FIRST ASSERTION. *For an arbitrary point $x \in I^n$, a trial point $x^{(i)}$ exist so near to x that*

$$\left|f_p\left(x^{(i)}\right) - f_p(x)\right| \leq d_\rho, \tag{4}$$

for all $1 \leq p \leq k$.

This assertion follows immediately from (3). Of course, the assertion is true when x is an efficient point also.

The trial point $x^{(i)}$ in (4) that corresponds to a point $x \in E$ is not necessarily approximately efficient. If we retain trial points from E_N only, a weaker statement can be made.

SECOND ASSERTION. *For an arbitrary $x \in E$, a trial point $x^{(s)} \in E_N$ exists satisfying the following requirement:*

$$\min_{1 \leq p \leq k} \left|f_p\left(x^{(s)}\right) - f_p(x)\right| \leq d_\rho. \tag{5}$$

PROOF. If the trial point $x^{(i)}$ satisfying (4) is in E_N, then clearly (5) is true with $x^{(s)} = x^{(i)}$. If $x^{(i)}$ is not in E_N, it is dominated by another trial point $x^{(s)} \in E_N$.

Since x is an efficient point inequalities

$$f_p\left(x^{(s)}\right) \leq f_p(x)$$

cannot be true for all $1 \leq p \leq k$ simultaneously, and for at least one index $p = q$,

$$f_q(x) \leq f_q\left(x^{(s)}\right) \leq f_q\left(x^{(i)}\right). \tag{6}$$

It follows from (4) and (6) that

$$f_q\left(x^{(s)}\right) - f_q(x) \leq d_\rho$$

and this implies (5). ∎

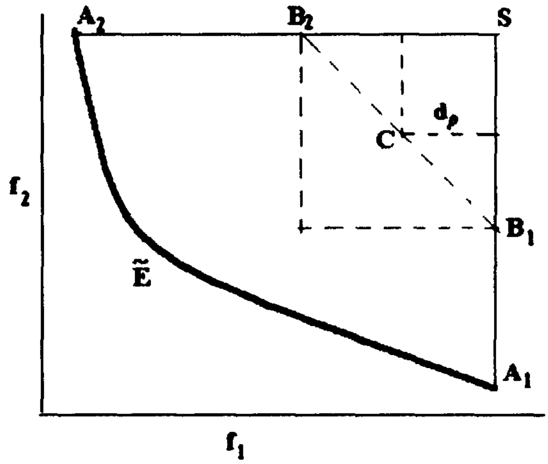

Figure 1. Proof of theorem: the contrary assumption.

4. CRITERIA SPACE

Functions $f_1(x), \dots, f_k(x)$ define a mapping of I^n into the k-dimensional *criteria space* $(f_1, \dots, f_k)$. The image of E in the criteria space will be denoted by $\tilde{E}$ and called *Pareto set* (or Pareto-Edgeworth set [5]). Similarly, the image of E_N will be called *approximate Pareto set* and denoted by $\tilde{E}_N$. And the image of I^n will be denoted by $\tilde{G}$.

Consider an arbitrary point $x^{(s)} \in E_N$ and its image $S = (f_1^s, \dots, f_k^s)$; here $f_p^s = f_p(x^{(s)})$. Assume that straight lines parallel to coordinate axes in the criteria space passing through S intersect $\tilde{E}$ in points $A_1, \dots, A_k$ (Figure 1).

The distance from S to $\tilde{E}$ will be measured in all k directions

$$\Delta_p(S) = |S\,A_p|, \qquad 1 \le p \le k.$$

THEOREM. *If the part of $\tilde{E}$ containing the points $A_1, \dots, A_k$ is a smooth convex hypersurface and the "pyramid" $SA_1 \dots A_k$ belongs to $\tilde{G}$, then*

$$\min_{1 \le p \le k} \Delta_p(S) \le 2d_\rho. \tag{7}$$

PROOF. We have to prove that at least one of the distances $\Delta_1(S), \dots, \Delta_k(S)$ does not exceed $2d_\rho$. Assume the contrary: all $\Delta_p(S) > 2d_\rho$, $1 \le p \le k$.

Then points B_p exist such that $|SB_p| = 2d_\rho$ and the k-dimensional cube with vertices S, $B_1, \dots,$ B_k is inside the "pyramid" (Figure 1). Let C be the centre of the cube and x_C its preimage in I^n. According to (4), a trial point $x^{(m)}$ must exist satisfying inequalities

$$\left| f_p\left(x^{(m)}\right) - f_p(x_C) \right| \le d_\rho, \qquad 1 \le p \le k. \tag{8}$$

However, it is easy to prove that such a point $x^{(m)}$ cannot exist! Indeed, let the image of $x^{(m)}$ be M, $M = (f_1(x^{(m)}), \dots, f_k(x^{(m)}))$. On one hand, M cannot be outside the cube because at least in one direction (that is, for one q), the difference $|f_q(x^{(m)}) - f_q(x_C)|$ would exceed d_ρ which contradicts (8). On the other hand, M cannot be inside the cube because $x^{(m)}$ would dominate $x^{(s)}$ whose image is S, however, $x^{(s)} \in E_N$. ∎

REMARK. A similar theorem for the case $k = 2$ was announced in [3]. However, there is a mistake in the proof and the factor 2 in (7) was lost.

APPROXIMATION ERROR. The theorem suggests the following measure of proximity of $\tilde{E}_N$ to $\tilde{E}$ (or E_N to E):

$$\Delta(N) = \max_s \min_{1 \le p \le k} \Delta_p(S). \tag{9}$$

5. TRIAL POINTS

An estimate of the lower bound for d_ρ was introduced in [4], namely,

$$c_\rho = \frac{1}{2} \max \left(\frac{s! L_{j_1} \dots L_{j_s}}{N} \right)^{1/s}; \tag{10}$$

the maximum in (10) is extended over all sets

$$1 \le j_1 < \dots < j_s \le n, \qquad s = 1, 2, \dots, n.$$

It was proved in [4] that

(i) for arbitrary points $x^{(1)}, \dots, x^{(N)}$

$$d_\rho \ge c_\rho;$$

(ii) for an arbitrary P_τ-net $x^{(1)}, \dots, x^{(N)}$ in I^n

$$d_\rho \le A c_\rho, \tag{11}$$

where $A = A(n, \tau)$ depends neither on N nor on $L_1, \dots, L_n$ (P_τ-nets are often called (t, m, s)-nets in base 2; here $t \equiv \tau$, $s \equiv n$-dimension, $m = \log_2 N$).

Clearly, c_ρ defines the best convergence rate of d_ρ as $N \to \infty$.

Traditional optimization theories consider only Lipscitz classes with equal Lipschitz constants $L_j = L$ for $1 \le j \le n$. In this case, (10) implies that

$$c_\rho = \frac{1}{2} (n!)^{1/n} L N^{-1/n}. \tag{12}$$

At large n, the order of convergence (12) is poor. However, if there are only t positive constants among the L_j, $t < n$, then $c_\rho \sim N^{-1/t}$ which can be much better than (12). One may expect that if the L_j are of different orders of magnitude, (10) will be a much more realistic estimate than (12).

In practical problems, the total number n of decision variables may be large, but individual objectives often depend heavily on a few of these variables and are not very sensitive to the others; therefore, most of the L_j are very small indeed.

QUASI-RANDOM TRIAL POINTS. As a rule in PSI, initial points $x^{(1)}, \dots, x^{(N)}$ of an LP_τ-sequence are used as trial points [1,2]. These sequences are often called (t, s)-sequences in base 2 (here $t \equiv \tau$, $s \equiv n$-dimension) or simply Sobol sequences. Initial sections of these sequences containing $N = 2^m$ points (m-integer) are P_τ-nets at all sufficiently large m. Therefore, it is advisable to monitor the convergence comparing results obtained at successive m when inequality (11) holds.

The most convenient subroutines for generating LP_τ-sequences were published in [6]. The languages are FORTRAN-77 and "C", $N < 2^{30}$. The published modifed direction numbers are for $n \le 51$, however, an extension to $n \le 370$ is available.

RANDOM TRIAL POINTS. For comparison, random trial points $x^{(1)}, x^{(2)}, \dots$ were used also. We have applied a pseudo-random number generator [7] to produce standard random numbers $\gamma_1, \gamma_2, \dots$ and defined

$$x^{(1)} = (\gamma_1, \dots, \gamma_n), \qquad x^{(2)} = (\gamma_{n+1}, \dots, \gamma_{2n}), \dots .$$

RECTANGULAR LATTICES. Consider a rectangular lattice consisting of $N = R^n$ points with Cartesian coordinates

$$x_1 = \frac{j_1 - 1/2}{R}, \dots, x_n = \frac{j_n - 1/2}{R},$$

where $j_1, \dots, j_n$ range independently over values $1, 2, \dots, R$. For this lattice, the ρ-dispersion d_ρ was computed in [4]:

$$d_\rho = \frac{1}{2} \sum_{j=1}^{n} L_j N^{-1/n}.$$

Comparing this value with (12), one can notice that rectangular lattices are almost optimal for Lipschitz classes with all $L_j = L$.

But this advantage disappears as soon as different constants $L_j$are considered, cf., [4,8]. Therefore, rectangular lattices should not be recommended for solving more-or-less general problems. Besides, the amount of trial points $N = R^n$ increases rapidly with increasing n so that in higher dimensions such lattices are impractical.

6. EXAMPLE

In order to reduce the amount of indices, we shall change our notations. Cartesian coordinates of a point in I^3 will be denoted by x, y, z.

Consider three objective functions

$$\begin{aligned} f &= (x-1)^2 + y^2 + \varepsilon z^2, \\ g &= x^2 + (y-1)^2 + \varepsilon z^2, \\ h &= x^2 + y^2 + \varepsilon (z-1)^2, \end{aligned} \tag{13}$$

that include a positive parameter ε. The multicriterial optimization problem is

$$f \to \min, \quad g \to \min, \quad h \to \min; \qquad (x, y, z) \in I^3. \tag{14}$$

The set of efficient points E for this problem is a triangle: the intersection of I^3 with the plane $x + y + z = 1$.

The last assertion can be proved in different ways. Maybe the simplest approach is a geometrical one: the objectives (13) can be interpreted as squared distances from the point (x, y, z) to the vertices $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. Clearly, every point (x, y, z) outside the triangle is dominated by its projection onto the triangle. Further, if we select two arbitrary points (x', y', z') and (x'', y'', z'') inside the triangle, then one of these points cannot dominate the other: e.g., if (x', y', z') is nearer than (x'', y'', z'') to two vertices then (x'', y'', z'') is nearer to the third vertex.

Formulas (13) define a mapping of I^3 into the criteria space (f, g, h). The Jacobian of the mapping is

$$\frac{\partial(f, g, h)}{\partial(x, y, z)} = 8\varepsilon(x + y + z - 1).$$

Clearly, on both sides of the plane $x + y + z = 1$, the mapping is univalent but the images of both parts are overlapping.

THE PARETO SET IN THE CRITERIA SPACE. Denote temporarily

$$w = x^2 + y^2 + \varepsilon z^2. \tag{15}$$

Then formulas (13) can be written as

$$f = w - 2x + 1, \qquad g = w - 2y + 1, \qquad h = w - 2\varepsilon z + \varepsilon. \tag{16}$$

From (16),

$$x = \frac{1}{2}(w - f + 1), \qquad y = \frac{1}{2}(w - g + 1), \qquad z = \frac{1}{2\varepsilon}(w - h + \varepsilon). \tag{17}$$

Substituting (17) into (15), we obtain a quadratic equation for w,

$$\left(2+\frac{1}{\varepsilon}\right)w^2-2w\left(f+g+\frac{h}{\varepsilon}-1\right)+(1-f)^2+(1-g)^2+\varepsilon\left(1-\frac{h}{\varepsilon}\right)^2=0.$$

Of the two solutions, the one with a $+$ sign corresponds to the half-space $x+y+z>1$, and the one with a $-$ sign corresponds to the half-space $x+y+z<1$. Hence, the image of the plane $x+y+z=1$ can be defined by the vanishing discriminant

$$\left(f+g+\frac{h}{\varepsilon}-1\right)^2-\left(2+\frac{1}{\varepsilon}\right)\left[(1-f)^2+(1-g)^2+\varepsilon\left(1-\frac{h}{\varepsilon}\right)^2\right]=0.$$

The final form of the last equation is

$$(1+\varepsilon)\left(f^2+g^2\right)+2h^2-2[\varepsilon fg+fh+gh+(1+\varepsilon)(f+g)+2\varepsilon h]+2(1+\varepsilon)^2=0. \tag{18}$$

The surface (18) is an elliptic paraboloid. Its vertex is the point

$$f=\frac{\varepsilon+1}{3},\qquad g=\frac{\varepsilon+1}{3},\qquad h=\frac{2}{3},$$

and its axis—the straight line

$$f=g=h+\frac{\varepsilon-1}{3},\qquad \text{at } h\geq\frac{2}{3}.$$

Of course, the paraboloid (18) is the image of the whole plane $x+y+z=1$, while the Pareto set $\tilde{E}$ is a finite part of (18) that corresponds to E.

7. COMPUTATION FORMULAS

Let $(x^{(s)},y^{(s)},z^{(s)})$ be an approximately efficient trial point and $S=(f_s,g_s,h_s)$ its image inside the paraboloid (18); here

$$f_s=f\left(x^{(s)},y^{(s)},z^{(s)}\right),\qquad g_s=g\left(x^{(s)},y^{(s)},z^{(s)}\right),\qquad h_s=h\left(x^{(s)},y^{(s)},z^{(s)}\right).$$

The three distances from S to $\tilde{E}$ can be found by solving three quadratic equations induced by (18),

(i)
$$A_f=1+\varepsilon;\qquad B_f=\varepsilon g_s+h_s+1+\varepsilon;$$
$$C_f=(1+\varepsilon)g_s^2+2h_s^2-2g_sh_s-2(1+\varepsilon)g_s-4\varepsilon h_s+2(1+\varepsilon)^2;$$
$$f_s^*=\frac{1}{A_f}\left(B_f-\sqrt{B_f^2-A_fC_f}\right);\qquad \Delta_f(S)=f_s-f_s^*;$$

(ii)
$$A_g=1+\varepsilon;\qquad B_g=\varepsilon f_s+h_s+1+\varepsilon;$$
$$C_g=(1+\varepsilon)f_s^2+2h_s^2-2f_sh_s-2(1+\varepsilon)f_s-4\varepsilon h_s+2(1+\varepsilon)^2;$$
$$g_s^*=\frac{1}{A_g}\left(B_g-\sqrt{B_g^2-A_gC_g}\right);\qquad \Delta_g(S)=g_s-g_s^*;$$

(iii)
$$A_h=2;\qquad B_h=f_s+g_s+2\varepsilon;$$
$$C_h=(1+\varepsilon)\left(f_s^2+g_s^2\right)-2\varepsilon f_sg_s-2(1+\varepsilon)(f_s+g_s)+2(1+\varepsilon)^2;$$
$$h_s^*=\frac{1}{A_h}\left(B_h-\sqrt{B_h^2-A_hC_h}\right);\qquad \Delta_h(S)=h_s-h_s^*.$$

Let $\Delta_s=\min[\Delta_f(S);\Delta_g(S);\Delta_h(S)]$.

The approximation error (9) for our example is

$$\Delta(N)=\max_s\Delta_s;$$

the maximum is over all N_0 approximately efficient trial points $x^{(s)}\in E_N$.

8. NUMERICAL RESULTS

Three different problems (14) were computed that correspond to $\varepsilon = 1$, $\varepsilon = 10$, and $\varepsilon = 0.01$. Of these, the problem with $\varepsilon = 1$ is the simplest one since it is symmetric in all variables x, y, z. In each case, three types of trial points were used: quasi-random, random, and rectangular lattices.

Figure 2 illustrates the convergence of $\tilde{E}_N$ (which is a discrete set of N_0 points) to $\tilde{E}$ as the quantity of trial points N increases. The exact boundary curve

$$f = u + v, \quad g = u - v, \qquad \text{where } u = \frac{1}{2}\left(v^2 + 1\right), \ -1 \le v \le 1,$$

was obtained from the equation $B_h^2 - A_h C_h = 0$. It does not depend on ε.

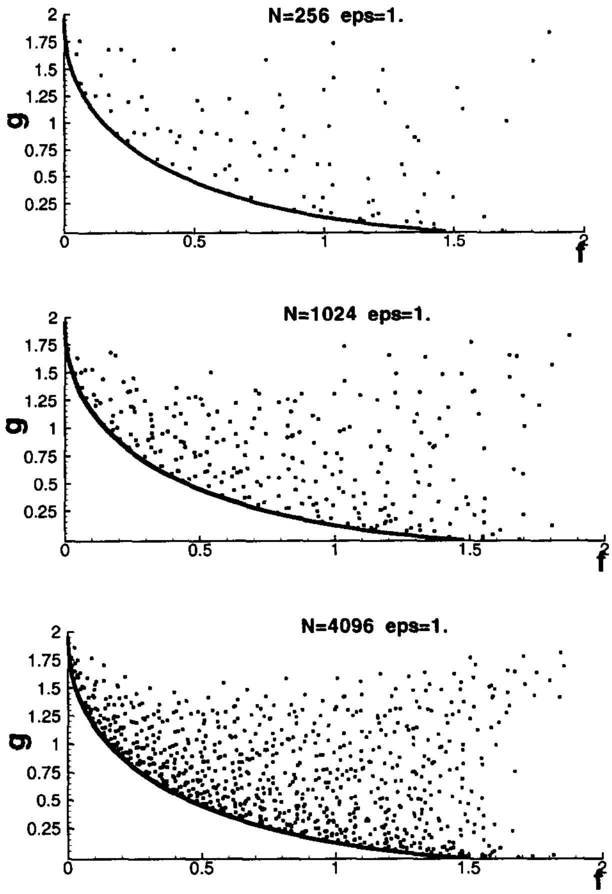

Figure 2. Projections of quasi-random approximately efficient points onto the (f, g)-plane at $N = 256$, $N = 1024$, and $N = 4096$ for eps $= 1$.

Figure 3 shows how different are the shapes of $\tilde{E}$ for the three problems considered. Here the exact boundary curve

$$f = u + v, \quad h = u - v, \qquad \text{where } u = \frac{v^2}{1+\varepsilon} + \frac{1+\varepsilon}{4}, \ -\frac{1+\varepsilon}{2} \le v \le \frac{1+\varepsilon}{2},$$

was obtained from the equation $B_g^2 - A_g C_g = 0$. The (g, h) projections of $\tilde{E}$ are identical with the (f, h) projections because (14) is symmetric in f and g.

The computed approximation errors $\Delta(N)$ are presented in Table 1 with crude rounding.

More informative is Figure 4 where the scale is logarithmic: $\log_{10} \Delta(N)$ versus $\log_{10} N$, so that a linear disposition of computed points means a power law convergence rate of $\Delta(N)$.

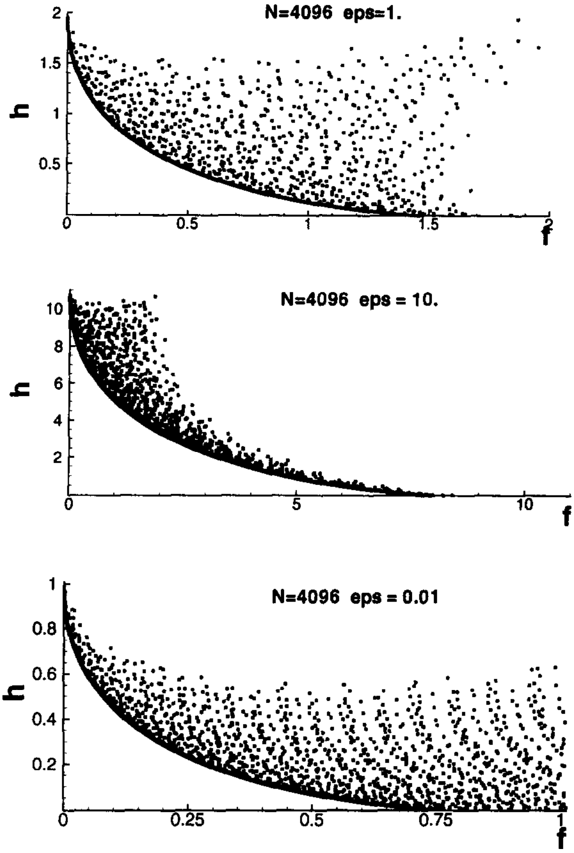

Figure 3. Projections of quasi-random approximately efficient points onto the (f, h)-plane at $N = 4096$ for different eps.

Table 1. Computed values of approximation errors $\Delta(N)$ (2.2^{-1} means $2.2 \cdot 10^{-1}$).

$\varepsilon = 1$					$\varepsilon = 10$					$\varepsilon = 0.01$				
N	Q-R $\Delta(N)$	RAND $\Delta(N)$	N	LAT $\Delta(N)$	N	Q-R $\Delta(N)$	RAND $\Delta(N)$	N	LAT $\Delta(N)$	N	Q-R $\Delta(N)$	RAND $\Delta(N)$	N	LAT $\Delta(N)$
2^7	2.2^{-1}	2.7^{-1}	5^3	7.3^{-2}	2^7	4.4^{-1}	8.0^{-1}	5^3	4.9^{-1}	2^7	1.9^{-2}	1.6^{-2}	5^3	1.6^{-2}
2^8	2.2^{-1}	2.2^{-1}	8^3	3.1^{-2}	2^8	3.7^{-1}	4.9^{-1}	8^3	5.0^{-1}	2^8	1.9^{-2}	1.6^{-2}	8^3	1.6^{-2}
2^9	1.3^{-1}	2.5^{-1}	12^3	1.6^{-2}	2^9	3.6^{-1}	4.9^{-1}	12^3	4.6^{-1}	2^9	1.8^{-2}	1.8^{-2}	12^3	1.7^{-2}
2^{10}	1.3^{-1}	1.6^{-1}	16^3	7.8^{-3}	2^{10}	3.6^{-1}	3.1^{-1}	16^3	4.4^{-1}	2^{10}	1.6^{-2}	1.6^{-2}	16^3	1.7^{-2}
2^{11}	1.2^{-1}	9.8^{-2}	20^3	5.3^{-3}	2^{11}	2.5^{-1}	2.8^{-1}	20^3	2.8^{-1}	2^{11}	1.6^{-2}	1.7^{-2}	20^3	1.8^{-2}
2^{12}	9.3^{-2}	9.6^{-2}	24^3	3.9^{-3}	2^{12}	1.8^{-1}	2.8^{-1}	24^3	2.2^{-1}	2^{12}	1.6^{-2}	1.7^{-2}	24^3	1.7^{-2}
2^{13}	7.0^{-2}	7.4^{-2}	28^3	2.7^{-3}	2^{13}	1.8^{-1}	1.8^{-1}	28^3	1.8^{-1}	2^{13}	1.6^{-2}	1.6^{-2}	28^3	1.7^{-2}
2^{14}	5.7^{-2}	6.0^{-2}	32^3	2.1^{-3}	2^{14}	1.3^{-1}	1.6^{-1}	32^3	1.5^{-1}	2^{14}	1.4^{-2}	1.6^{-2}	32^3	1.6^{-2}
2^{15}	4.7^{-2}	4.4^{-2}	40^3	1.3^{-3}	2^{15}	1.1^{-1}	1.0^{-1}			2^{15}	1.3^{-2}	1.5^{-2}	40^3	1.6^{-2}
2^{16}	3.5^{-2}	3.8^{-2}								2^{16}	1.2^{-2}	1.2^{-2}		
2^{17}	3.1^{-2}	3.1^{-2}								2^{17}	1.0^{-2}			

For the symmetric problem ($\varepsilon = 1$), the performances of quasi-random and random trial points (circles and triangles, respectively,) are rather similar. For all N, they are near to the straight line

$$\log_{10} \Delta(N) = -\frac{1}{3} \log_{10} N + 0.16,$$

so that $\Delta(N)$ is of the order $N^{-1/3}$. This agrees with (12) at $n = 3$.

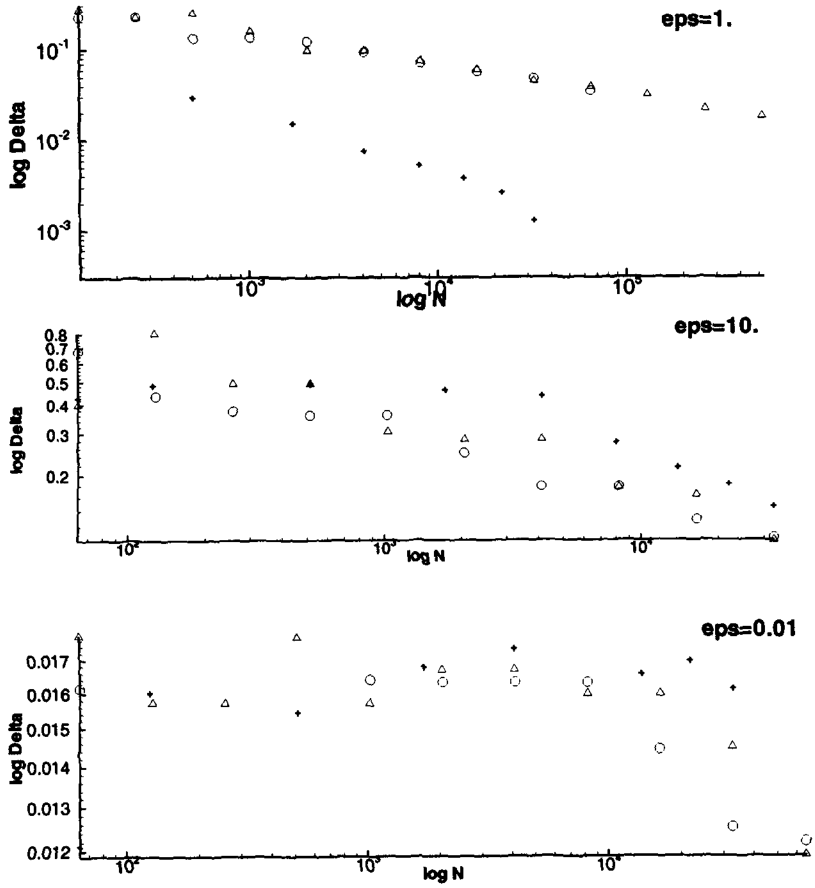

Figure 4. Approximation errors $\Delta(N)$ = Delta for different problems, computed with quasi-random points (circles), random points (triangles), or rectangular lattices (crosses).

Surprisingly, the performance of rectangular lattices (crosses) is much more effective, approximately

$$\log_{10} \Delta(N) = -\frac{2}{3} \log_{10} N + 0.32.$$

However, the advantage of rectangular lattices holds only for symmetric problems: for nonsymmetric cases ($\varepsilon = 10$ and $\varepsilon = 0.01$) rectangular lattices perform poorer than quasi-random or random trial points.

For the problem with $\varepsilon = 10$, there is a clear linear disposition of circles at large N and the power law is again $N^{-1/3}$, while for the problem with $\varepsilon = 0.01$, the number of trial points N seems to be insufficient and the asymptotics of $\Delta(N)$ is not clear.

All three objective functions (13) satisfy the general Lipschitz condition (2) with $L_1 = L_2 = 2$, $L_3 = 2\epsilon$. From (10), we can compute c_p values and suggest that quasi-random trial points produce approximation errors $\Delta(N)$ of the same order as c_p.

Let $N = 4096$. For all three computed problems

$$c_p = \frac{1}{2} \left(\frac{48\varepsilon}{N}\right)^{1/3}.$$

At $\varepsilon = 1, 10, 0.01$, the values are

$$c_p = 0.11, 0.25, 0.025.$$

The corresponding figures from Table 1 are

$$\Delta(N) = 0.09, 0.18, 0.016.$$

Clearly, the agreement is more than satisfactory.

Finally, it can be noticed that as N increased, the amount N_0 of approximately efficient points increased as N^β with β varying from 0.71 to 0.93 for different ε and different trial points.

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